X Chapter: Quardatic Equation Solved Question and Self Evaluation Question part-1

Nature of roots of a quadratic equation

The roots of the equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac > 0$, we get two distinct real roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac = 0$, then the equation has two equal roots $x = \frac{-b}{2a}$.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Therefore there is no real root for the given quadratic equation.

Determine the nature of roots of the following quadratic equations

(i) $x^2 - 11x - 10 = 0$ (ii) $4x^2 - 28x + 49 = 0$ (iii) $2x^2 + 5x + 5 = 0$

Solution For $ax^2 + bx + c = 0$, the discriminant, $\Delta = b^2 - 4ac$.

(i) Here, $a = 1$; $b = -11$ and $c = -10$.

Now, the discriminant is $\Delta = b^2 - 4ac$

$= (-11)^2 - 4(1)(-10) = 121 + 40 = 161$

Thus, $\Delta > 0$. Therefore, the roots are real and unequal.

(ii) Here, $a = 4$, $b = -28$ and $c = 49$.

Now, the discriminant is $\Delta = b^2 - 4ac$

$= (-28)^2 - 4(4)(49) = 0$

Since $\Delta = 0$, the roots of the given equation are real and equal.

(iii) Here, $a = 2$, $b = 5$ and $c = 5$.

Now, the discriminant $\Delta = b^2 - 4ac = (5)^2 - 4(2)(5) = 25 - 40 = -15$

Since $\Delta < 0$, the equation has no real roots.

Prove that the roots of the equation $(a-b+c)x^2 + 2(a-b)x + (a-b-c) = 0$ are rational numbers for all real numbers a and b and for all rational c.

Solution Let the given equation be of the form $Ax^2 + Bx + C = 0$. Then,

$A = a - b + c$, $B = 2(a-b)$ and $C = a - b - c$.

Now, the discriminant of $Ax^2 + Bx + c = 0$ is

$$B^2 - 4AC = [2(a-b)]^2 - 4(a-b+c)(a-b-c)$$
$$= 4(a-b)^2 - 4[(a-b)+c][(a-b)-c]$$
$$= 4(a-b)^2 - 4[(a-b)^2 - c^2]$$
$$\Delta = 4(a-b)^2 - 4(a-b)^2 + 4c^2 = 4c^2, \text{ a perfect square.}$$

Therefore, $\Delta > 0$ and it is a perfect square.

Hence, the roots of the given equation are rational numbers.

Find the values of k so that the equation $x^2 - 2x(1+3k) + 7(3+2k) = 0$ has real and equal roots.

Solution The given equation is $x^2 - 2x(1+3k) + 7(3+2k) = 0$. (1)

Let the equation (1) be in the form $ax^2 + bx + c = 0$

Here, $a = 1$, $b = -2(3k+1)$, $c = 7(3+2k)$.

Now, the discriminant is $\Delta = b^2 - 4ac$

$$= (-2(3k+1))^2 - 4(1)(7)(3+2k)$$

$$= 4(9k^2 + 6k + 1) - 28(3+2k) = 4(9k^2 - 8k - 20)$$

Given that the equation has equal roots. Thus, $\Delta = 0$

$$\Longrightarrow 9k^2 - 8k - 20 = 0$$

$$\Longrightarrow (k-2)(9k+10) = 0$$

Thus, $k = 2, -\frac{10}{9}$.

Self Evaluation

1. Determine the nature of the roots of the equation.

   (i) $x^2 - 8x + 12 = 0$ (ii) $2x^2 - 3x + 4 = 0$

   (iii) $9x^2 + 12x + 4 = 0$ (iv) $3x^2 - 2\sqrt{6}x + 2 = 0$

   (v) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$ (vi) $(x-2a)(x-2b) = 4ab$

2. Find the values of k for which the roots are real and equal in each of the following equations.

   (i) $2x^2 - 10x + k = 0$ (ii) $12x^2 + 4kx + 3 = 0$

   (iii) $x^2 + 2k(x-2) + 5 = 0$ (iv) $(k+1)x^2 - 2(k-1)x + 1 = 0$

3. Show that the roots of the equation $x^2 + 2(a+b)x + 2(a^2+b^2) = 0$ are unreal.

4. Show that the roots of the equation $3p^2x^2 - 2pqx + q^2 = 0$ are not real.

5. If the roots of the equation $(a^2+b^2)x^2 - 2(ac+bd)x + c^2 + d^2 = 0$, where $ad - bc \neq 0$, are equal, prove that $\frac{a}{b} = \frac{c}{d}$.

6. Show that the roots of the equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real and they cannot be equal unless $a = b = c$.

7. If the equation $(1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, then prove that $c^2 = a^2(1+m^2)$.

Relations between roots and coefficients of a quadratic equation

Therefore, if α, β are the roots of $ax^2 + bx + c = 0$, then

(i) the sum of the roots, $\alpha + \beta = -\frac{b}{a}$

(ii) the product of roots, $\alpha\beta = \frac{c}{a}$

Formation of quadratic equation when roots are given

Let α and β be the roots of a quadratic equation.

Then $(x-\alpha)$ and $(x-\beta)$ are factors.

$$\therefore \quad (x-\alpha)(x-\beta)=0$$

$$\Longrightarrow \quad x^2-(\alpha+\beta)x+\alpha\beta=0$$

That is, $x^2-(\text{sum of roots})x+\text{product of roots}=0$

If one of the roots of the equation $3x^2-10x+k=0$ is $\frac{1}{3}$, then find the other root and also the value of k.

Solution The given equation is $3x^2-10x+k=0$.

Let the two roots be α and β.

$$\therefore \quad \alpha+\beta=\frac{-(-10)}{3}=\frac{10}{3} \qquad (1)$$

Substituting $\alpha=\frac{1}{3}$ in (1) we get $\beta=3$

Also, $\alpha\beta=\frac{k}{3}$, $\Longrightarrow k=3$

Thus, the other root $\beta=3$ and the value of $k=3$.

If the sum and product of the roots of the quadratic equation $ax^2-5x+c=0$ are both equal to 10, then find the values of a and c.

Solution The given equation is $ax^2-5x+c=0$.

Sum of the roots, $\frac{5}{a}=10$, $\Longrightarrow a=\frac{1}{2}$

Product of the roots, $\frac{c}{a}=10$

$\Longrightarrow$ $c=10a=10\times\frac{1}{2}=5$

Hence, $a=\frac{1}{2}$ and $c=5$

If α and β are the roots of the equation $2x^2-3x-1=0$, find the values of

(i) $\alpha^2+\beta^2$ (ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (iii) $\alpha-\beta$ if $\alpha>\beta$ (iv) $\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)$

(v) $\left(\alpha+\frac{1}{\beta}\right)\left(\frac{1}{\alpha}+\beta\right)$ (vi) $\alpha^4+\beta^4$ (vii) $\frac{\alpha^3}{\beta}+\frac{\beta^3}{\alpha}$

Solution Given equation is $2x^2-3x-1=0$

Let the given equation be written as $ax^2+bx+c=0$

Then, $a=2$, $b=-3$, $c=-1$. Given α and β are the roots of the equation.

$$\therefore \alpha+\beta=\frac{-b}{a}=\frac{-(-3)}{2}=\frac{3}{2} \text{ and } \alpha\beta=-\frac{1}{2}$$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{3}{2}\right)^2-2\left(-\frac{1}{2}\right)=\frac{9}{4}+1=\frac{13}{4}$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(\frac{3}{2}\right)^2-2\left(-\frac{1}{2}\right)}{-\frac{1}{2}}=\frac{13}{4}\times(-2)=-\frac{13}{2}$

(iii) $\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\left[\left(\frac{3}{2}\right)^2-4\times\left(-\frac{1}{2}\right)\right]^{\frac{1}{2}}=\left(\frac{9}{4}+2\right)^{\frac{1}{2}}=\frac{\sqrt{17}}{2}$

(iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\frac{27}{8}+\frac{9}{4}}{\frac{-1}{2}}=-\frac{45}{4}$

(v) $\left(\alpha+\frac{1}{\beta}\right)\left(\frac{1}{\alpha}+\beta\right)=\frac{(\alpha\beta+1)(1+\alpha\beta)}{\alpha\beta}$

$$=\frac{(1+\alpha\beta)^2}{\alpha\beta}=\frac{\left(1-\frac{1}{2}\right)^2}{-\frac{1}{2}}=-\frac{1}{2}$$

(vi) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$=\left(\frac{13}{4}\right)^2-2\left(-\frac{1}{2}\right)^2=\left(\frac{169}{16}-\frac{1}{2}\right)=\frac{161}{16}.$$

(vii) $\frac{\alpha^3}{\beta}+\frac{\beta^3}{\alpha}=\frac{\alpha^4+\beta^4}{\alpha\beta}=\left(\frac{161}{16}\right)\left(-\frac{2}{1}\right)=-\frac{161}{8}.$

Form the quadratic equation whose roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

Solution Given roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

$\therefore$ Sum of the roots $=7+\sqrt{3}+7-\sqrt{3}=14$.

Product of roots $=(7+\sqrt{3})(7-\sqrt{3})=(7)^2-(\sqrt{3})^2=49-3=46$.

The required equation is $x^2-(\text{sum of the roots})x+(\text{product of the roots})=0$

Thus, the required equation is $x^2-14x+46=0$

If α and β are the roots of the equation $3x^2-4x+1=0$, form a quadratic equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.

Solution Since α,β are the roots of the equation $3x^2-4x+1=0$,

we have $\alpha+\beta=\frac{4}{3}$, $\alpha\beta=\frac{1}{3}$

Now, for the required equation, the sum of the roots $=\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)=\frac{\alpha^3+\beta^3}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(\frac{4}{3}\right)^3-3\times\frac{1}{3}\times\frac{4}{3}}{\frac{1}{3}}=\frac{28}{9}$$

Also, product of the roots $=\left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right)=\alpha\beta=\frac{1}{3}$

$\therefore$ The required equation is $x^2-\frac{28}{9}x+\frac{1}{3}=0$ or $9x^2-28x+3=0$

Self Evaluation Question Bank

1. Find the sum and the product of the roots of the following equations.

   (i) $x^2 - 6x + 5 = 0$ (ii) $kx^2 + rx + pk = 0$

   (iii) $3x^2 - 5x = 0$ (iv) $8x^2 - 25 = 0$

2. Form a quadratic equation whose roots are

   (i) 3 , 4 (ii) $3 + \sqrt{7}, 3 - \sqrt{7}$ (iii) $\frac{4+\sqrt{7}}{2}, \frac{4-\sqrt{7}}{2}$

3. If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, then find the values of

   (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii) $\alpha - \beta$ (iii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

4. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\alpha^2 + \beta^2$.

5. If α, β are the roots of $2x^2 - 3x - 5 = 0$, form a equation whose roots are α^2 and β^2.

6. If α, β are the roots of $x^2 - 3x + 2 = 0$, form a quadratic equation whose roots are $-\alpha$ and $-\beta$.

7. If α and β are the roots of $x^2 - 3x - 1 = 0$, then form a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

8. If α and β are the roots of the equation $3x^2 - 6x + 1 = 0$, form an equation whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\alpha^2\beta, \beta^2\alpha$ (iii) $2\alpha + \beta, 2\beta + \alpha$

9. Find a quadratic equation whose roots are the reciprocal of the roots of the equation $4x^2 - 3x - 1 = 0$.

10. If one root of the equation $3x^2 + kx - 81 = 0$ is the square of the other, find k.

11. If one root of the equation $2x^2 - ax + 64 = 0$ is twice the other, then find the value of a

12. If α and β are the roots of $5x^2 - px + 1 = 0$ and $\alpha - \beta = 1$, then find p.

If one zero of the polynomial $p(x) = (k+4)x^2 + 13x + 3k$ is reciprocal of the other, then k is equal to

(A) 2 (B) 3 (C) 4 (D) 5

The sum of two zeros of the polynomial $f(x) = 2x^2 + (p+3)x + 5$ is zero, then the value of p is

(A) 3 (B) 4 (C) –3 (D) –4

The square root of $49(x^2 - 2xy + y^2)^2$ is

(A) $7|x - y|$ (B) $7(x+y)(x-y)$ (C) $7(x+y)^2$ (D) $7(x-y)^2$

The square root of $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

(A) $|x + y - z|$ (B) $|x - y + z|$ (C) $|x + y + z|$ (D) $|x - y - z|$

If $ax^2 + bx + c = 0$ has equal roots, then c is equal

(A) $\frac{b^2}{2a}$ (B) $\frac{b^2}{4a}$ (C) $-\frac{b^2}{2a}$ (D) $-\frac{b^2}{4a}$

If $x^2 + 5kx + 16 = 0$ has no real roots, then

(A) $k > \frac{8}{5}$ (B) $k > -\frac{8}{5}$ (C) $-\frac{8}{5} < k < \frac{8}{5}$ (D) $0 < k < \frac{8}{5}$

If $b = a + c$, then the equation $ax^2 + bx + c = 0$ has

(A) real roots (B) no roots (C) equal roots (D) no real roots