**Class 09 Chapter Number System Solved Problems**

**I.** Show that between two distinct rational numbers $a$ and $b$ there exists another rational number.

**Proof:** As $a$ and $b$ are two distinct rational numbers so either $a < b$ or $a > b$.

Without any loss of generality, we assume that $a < b$.

$$\therefore \quad \frac{a+b}{2} > \frac{a+a}{2} = a$$

Thus $\quad a < \frac{a+b}{2}$ .....(i)

And $\quad \frac{a+b}{2} < \frac{b+b}{2} = b$ .....(ii)

From ($i$) and ($ii$),

$$a < \frac{a+b}{2} < b$$

Thus a rational number $\frac{a+b}{2}$ lies between $a$ and $b$.

Hence proved.

**Note:** $\frac{a+b}{2}$ is known as *arithmetic mean* between $a$ and $b$.

The arithmetic mean between $a$ and $\frac{a+b}{2}$ is $\frac{a+\frac{a+b}{2}}{2} = \frac{3a+b}{4}$ which is a rational number and it lies between $a$ and $\frac{a+b}{2}$.

Thus $\quad a < \frac{3a+b}{4} < \frac{a+b}{2} < b.$

Also, the arithmetic mean between $a$ and $\frac{3a+b}{4}$ is $\frac{a+\frac{3a+b}{4}}{2} = \frac{7a+b}{8}$ which is a rational number between $a$ and $\frac{3a+b}{4}$.

Thus $a < \frac{7a+b}{8} < \frac{3a+b}{4} < \frac{a+b}{2} < b.$

Hence between two rational numbers there are infinitely many rational numbers.

**II.** Show that between two distinct rational numbers $a$ and $b$ there are infinitely many rational numbers.

**Proof:** Without any loss of generality, we assume that $a < b$.

$\therefore \quad b - a > 0$

Now, if $n$ rational numbers between $a$ and $b$ are required, then divide $(b - a)$ into $(n + 1)$ equal parts such that each part is $d = \frac{b-a}{n+1}$.

Then rational numbers situated at an equal interval from $a$ are $a + d$, $a + 2d$, $a + 3d$, ...., $a + nd$.

Thus between $a$ and $b$, the $n$ rational numbers are

$$a+\frac{b-a}{n+1},\ a+\frac{2(b-a)}{n+1},\ a+\frac{3(b-a)}{n+1},\ \ldots\ldots,\ a+\frac{n(b-a)}{n+1}.$$

If $n$ is increased indefinitely, then $\frac{b-a}{n+1}$ diminishes indefinitely, and there will be infinitely many rational numbers between $a$ and $b$.

*Prove that $\sqrt{2}$ is not a rational number.*

**Proof:** If possible, let $\sqrt{2}$ be a rational number and in its simplest form let $\sqrt{2} = p/q$, where $p$ and $q$ are integers having no common factor and $q \neq 0$.

Then $\quad \sqrt{2} = p/q \;\Rightarrow\; p^2 = 2q^2 \qquad$ ....(i)

As RHS of this equation is even so $p^2$ is even. Hence $p$ is also even. Let $p = 2m$ where $m \in Z$.

$\therefore \quad (2m)^2 = 2q^2 \qquad$ from $(i)$

or $\quad 4m^2 = 2q^2$

or $\quad q^2 = 2m^2 \qquad$ ....(ii)

RHS of $(ii)$ is even. So $q^2$ is even. Hence $q$ is even. Thus $p$ and $q$ are both even whose common factor is 2 which contradicts our hypothesis that $p$ and $q$ have no common factor.

Hence $\sqrt{2}$ is not a rational number.

## Define Irrational numbers

An irrational number when represented in decimal is a non-terminating non-recurring decimal.

***Example:*** $\sqrt{2} = 1.414213......$ $\sqrt{3} = 1.73205......$

Other examples of irrational numbers are,

1.64030030003....... ; 1.64272272227.......,

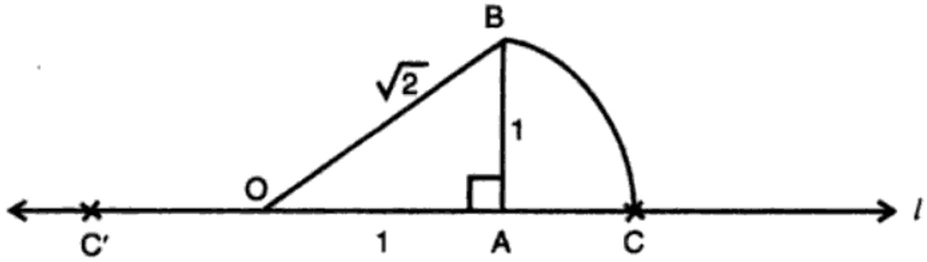
0.1210010001......... ; 0.12353353335.......,

0.51010010001....... ; 7.323323332........... .

## Representation of Irrational Numbers on Number Line

### (i) Representation of $\sqrt{2}$

Let O represent 0 (zero) and A represent 1 on the number line $l$.

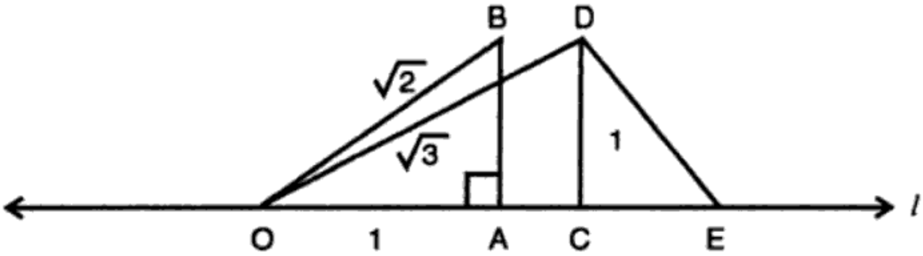



Draw a right angled $\Delta OBA$, $\angle A = 90°$ and OA = AB = 1 unit.

Then $OB = \sqrt{2}$. Taking O as centre and OB radius, draw an arc to cut the number line $l$ at C. Then $OC = \sqrt{2}$. Thus the point C represents $\sqrt{2}$. Corresponding to C, there is a point C' to the left of O which obviously represents $-\sqrt{2}$.

### (ii) Representation of $\sqrt{3}$

Let O represent 0 (zero) and A represent 1 on the number line $l$.




Draw a right angled $\Delta OAB$ right angled at A and OA = AB = 1. Then $OB = \sqrt{2}$. Taking O as centre and OB radius draw an arc to cut the number line $l$ at C. Then C represents $\sqrt{2}$. Now draw a right angled triangle right angled at C and CD = 1.

Then $OC = \sqrt{2}$, CD = 1

$\therefore$ By Pythagoras theorem, $OD^2 = OC^2 + CD^2$

or $OD^2 = (\sqrt{2})^2 + 1^2 = 2 + 1 = 3$

converting recurring decimals into rational number of form $p/q$.

***Example 1:*** Convert $0.\dot{6}$ into the form $p/q$

***Solution:*** Let $x = 0.\dot{6}$

$\Rightarrow \quad x = 0.666 \ldots$ .....(i)

Multiply both sides by 10, then $10x = 6.666 \ldots$ .....(ii)

Subtract (i) from (ii) $10x - x = 6.666 \ldots - 0.666 \ldots$

$\therefore \quad 9x = 6$

$\Rightarrow \quad x = \frac{6}{9} = \frac{2}{3}$

$\left[\textbf{Note: } 0\dot{6} = \frac{6}{9}\right]$

***Example 2:*** Convert $0.2\dot{7}$ into the form $p/q$

***Solution:*** Let $x = 0.2\dot{7} = 0.2777 \ldots$ .....(i)

Multiply both sides of (*i*) by 10, then $10x = 2.777 \ldots$ .....(ii)

Again, multiply both sides of (*i*) by 100, then $100x = 27.777 \ldots$ .....(iii)

Subtract (ii) from (iii)

$100x - 10x = 27.777 \ldots - 2.777 \ldots$

$\Rightarrow 90x = 27 - 2 = 25$

$\therefore x = \frac{25}{90} = \frac{5}{18}$

$\left[\textbf{Note: } 0.2\dot{7} = \frac{27-2}{90} = \frac{25}{90} = \frac{5}{18}\right]$

Prove that (i) $\sqrt{3}$ (ii) $\sqrt{5}$ are irrational numbers.

***Solution:***

(i) If possible, let $\sqrt{3}$ be a rational number

$\therefore 1^2 < 3 < 2^2 \qquad \therefore 1 < \sqrt{3} < 2$

Thus $\sqrt{3}$ lies between 1 and 2. Hence it is not an integer. Now let $\sqrt{3} = p/q$ where $p$ and $q$ have no common factor, $p$ and $q$ are integers and $q \neq 1$

$\therefore \quad 3 = p^2/q^2$

or $\quad 3q = p^2/q$. .....(i)

As $p$ and $q$ have no common factor, so $p^2$ and $q$ also have no common factor. Thus $p^2/q$ is a fraction. But $3q$ is an integer.

$\therefore$ Integer = fraction, which is impossible.

Thus $\sqrt{3}$ is not a rational number.

Hence $\sqrt{3}$ is an irrational number.

(ii) If possible, let $\sqrt{5}$ be a rational number.

$\because 2^2 < 5 < 3^2 \quad \therefore 2 < \sqrt{5} < 3$

Thus $\sqrt{5}$ lies between 2 and 3.

But there is no integer between 2 and 3.

Hence $\sqrt{5}$ is not an integer.

Let $\sqrt{5} = p/q$, where $p$ and $q$ are integers and have no common factor and $q \neq 1$.

$\therefore \quad 5 = p^2/q^2$

or $\quad 5q = p^2/q$ .....(i)

As $p$ and $q$ have no common factors, so $p^2$ and $q$ also have no common factor. Thus $p^2/q$ is a fraction.

But $5q$ is an integer.

$\therefore$ Integer = fraction, which is impossible.

Thus $\sqrt{5}$ is not a rational number.

Hence $\sqrt{5}$ is an irrational number.

Prove that $\sqrt{5} - \sqrt{3}$ is an irrational number.

***Solution:*** If possible, let $\sqrt{5} - \sqrt{3}$ be a rational number $p/q$.

Thus $\frac{p}{q} = \sqrt{5} - \sqrt{3}$, where $p$ and $q$ are integers having no common factor and $q \neq 0$.

$$\therefore \left(\frac{p}{q}\right)^2 = (\sqrt{5} - \sqrt{3})^2 = 5 + 3 - 2\sqrt{5} \times \sqrt{3}$$

or $\frac{p^2}{q^2} = 8 - 2\sqrt{15}$ .....(i)

As $\frac{p}{q} \in Q$

$\therefore \frac{p^2}{q^2} \in Q$ , because $p^2 = p \times p$ and $q^2 = q \times q$.

Thus LHS of (i) is rational but its RHS is irrational.

$\therefore$ Rational number = Irrational number, which is impossible.

Thus assumption $\sqrt{5} - \sqrt{3}$ as a rational number is not correct.

Hence $\sqrt{5} - \sqrt{3}$ is an irrational number. **Proved**

Determine three irrational numbers between 0.5 and 0.52.

***Solution:*** Take any number 0.51 which lies between 0.5 and 0.52. Now, the following are any three irrational numbers between 0.5 and 0.52.

0.5101001000100001......

0.5103003000300003......

0.5124224222422224......

Determine two irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

***Solution:***

**First Method:** We know that $\sqrt{2} = 1.4142.....$

and $\sqrt{3} = 1.7321.....$

Take any number between 1.4142.... and 1.7321 say 1.64.

Now two irrational numbers may be written as

1.64030030003.....

and 1.64272272227.....

[**Note:** Infinite number of irrational numbers between $\sqrt{2}$ and $\sqrt{3}$ can be determined].

**Second Method:** Here two irrational numbers are required.

Hence $n = 2$, $a = \sqrt{2}$ and $b = \sqrt{3}$

$$\therefore \quad d = \frac{b-a}{n+1} = \frac{\sqrt{3}-\sqrt{2}}{3}.$$

Then the required irrational numbers are $a + d$ and $a + 2d$.

or $\sqrt{2} + \frac{\sqrt{3}-\sqrt{2}}{3}$ and $\sqrt{2} + \frac{2(\sqrt{3}-\sqrt{2})}{3}$

or $\frac{2\sqrt{2}+\sqrt{3}}{3}$ and $\frac{\sqrt{2}+2\sqrt{3}}{3}$ are irrational numbers.

Determine any two rational numbers between $\sqrt{5}$ and $\sqrt{7}$.

***Solution:*** We know that $\sqrt{5} = 2.236.....$ $\sqrt{7} = 2.6458.....$

Now out of the infinite rational numbers between 2.236..... and 2.6458..... we may take any two, say 2.4 and 2.5.

Thus $2.4 = \frac{24}{10}$ and $2.5 = \frac{25}{10}$

e.g. $\frac{12}{5}$ and $\frac{5}{2}$ are any two rational numbers between $\sqrt{5}$ and $\sqrt{7}$.