Class-X Maths Introduction to Trigonometry Solved Problems

Example 1. *If A is an acute angle and tan A = $\frac{12}{5}$, find all other trigonometric ratios of the angle A (using trigonometric identities).*

Solution. Given tan A = $\frac{12}{5}$

$\Rightarrow \quad \cot A = \frac{1}{\tan A} = \frac{5}{12};$

$\sec^2 A = 1 + \tan^2 A = 1 + \left(\frac{12}{5}\right)^2 = \frac{25+144}{25} = \frac{169}{25}$

$\Rightarrow \quad \sec A = \frac{13}{5}$ (∵ sec A is +ve)

$\Rightarrow \quad \cos A = \frac{1}{\sec A} = \frac{5}{13};$

$\sin A = \frac{\sin A}{\cos A}.\cos A = \tan A \cos A = \frac{12}{5} \times \frac{5}{13} = \frac{12}{13}$

$\Rightarrow \quad \text{cosec } A = \frac{1}{\sin A} = \frac{13}{12}.$

Hence, $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\cot A = \frac{5}{12}$, $\sec A = \frac{13}{5}$, $\text{cosec } A = \frac{13}{12}$.

Example 2. *Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.* (NCERT)

Solution. $\cot A = \frac{1}{\tan A}$; $\text{cosec}^2 A = 1 + \cot^2 A \Rightarrow \text{cosec } A = \sqrt{1+\cot^2 A}$

$\Rightarrow \quad \sin A = \frac{1}{\text{cosec } A} = \frac{1}{\sqrt{1+\cot^2 A}}$; $\sec^2 A = 1 + \tan^2 A = 1 + \left(\frac{1}{\cot A}\right)^2 = \frac{\cot^2 A + 1}{\cot^2 A}$

$\Rightarrow \quad \sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$

Hence, $\sin A = \frac{1}{\sqrt{1+\cot^2 A}}$, $\sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}$, $\tan A = \frac{1}{\cot A}$.

Example 3. *Write all the other trigonometric ratios of ∠A in terms of sec A.* (NCERT)

Solution. $\cos A = \frac{1}{\sec A}$; $\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{1}{\sec A}\right)^2 = \frac{\sec^2 A - 1}{\sec^2 A}$

$\Rightarrow \quad \sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$; $\tan^2 A = \sec^2 A - 1 \Rightarrow \tan A = \sqrt{\sec^2 A - 1}$;

$\cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{\sec^2 A - 1}}$; $\text{cosec } A = \frac{1}{\sin A} = \frac{\sec A}{\sqrt{\sec^2 A - 1}}.$

Hence, $\sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \frac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$,

$\cot A = \frac{1}{\sqrt{\sec^2 A - 1}}$, $\text{cosec } A = \frac{\sec A}{\sqrt{\sec^2 A - 1}}.$

Example 4. *Given A is an acute angle and 13 sin A = 5, evaluate* $\frac{5\sin A - 2\cos A}{\tan A}$.

Solution. Given $13 \sin A = 5 \Rightarrow \sin A = \frac{5}{13}$.

We know that $\cos^2 A = 1 - \sin^2 A$

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$\Rightarrow \quad \cos^2 A = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{169-25}{169} = \frac{144}{169}$

$\Rightarrow \quad \cos A = \frac{12}{13}$ (as A is an acute angle, cos A is +ve)

$\tan A = \frac{\sin A}{\cos A} = \frac{5}{13} \times \frac{13}{12} = \frac{5}{12}$.

$\therefore \quad \frac{5\sin A - 2\cos A}{\tan A} = \frac{5 \times \frac{5}{13} - 2 \times \frac{12}{13}}{\frac{5}{12}} = \frac{\frac{1}{13}}{\frac{5}{12}} = \frac{1}{13} \times \frac{12}{5} = \frac{12}{65}$.

Example 5. *If* $\tan\theta = \frac{1}{\sqrt{5}}$, *find the value of* $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$.

Solution. Given $\tan\theta = \frac{1}{\sqrt{5}} \Rightarrow \cot\theta = \sqrt{5}$ $\left(\because \cot\theta = \frac{1}{\tan\theta}\right)$

Now $\quad \sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1 + \frac{1}{5} = \frac{6}{5}$

and $\text{cosec}^2\theta = 1 + \cot^2\theta = 1 + (\sqrt{5})^2 = 1 + 5 = 6$.

$\therefore \quad \frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = \frac{6 - \frac{6}{5}}{6 + \frac{6}{5}} = \frac{30-6}{30+6} = \frac{24}{36} = \frac{2}{3}$.

Example 6. *Evaluate the following:*

(*i*) $\frac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$ *(NCERT)* (*ii*) $\sin 25° \cos 65° + \cos 25° \sin 65°$ *(NCERT)*

Solution. (*i*) As $\sin 63° = \sin(90° - 27°) = \cos 27°$ and

$\cos 73° = \cos(90° - 17°) = \sin 17°$,

$\therefore \quad \frac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°} = \frac{\cos^2 27° + \sin^2 27°}{\cos^2 17° + \sin^2 17°}$

$= \frac{1}{1}$ $(\because \sin^2 A + \cos^2 A = 1)$

$= 1$.

(*ii*) $\sin 25° \cos 65° + \cos 25° \sin 65°$

$= \sin 25° \cos(90° - 25°) + \cos 25° \sin(90° - 25°)$

$= \sin 25° \sin 25° + \cos 25° \cos 25°$

$= \sin^2 25° + \cos^2 25°$

$= 1$ $(\because \sin^2 A + \cos^2 A = 1)$

Example 7. *Show that* $\frac{\cos^2(45° + \theta) + \cos^2(45° - \theta)}{\tan(60° + \theta)\tan(30° - \theta)} = 1$. *(NCERT Exemplar)*

Solution. As $\cos(45° - \theta) = \cos(90° - (45° + \theta)) = \sin(45° + \theta)$

and $\tan(30° - \theta) = \tan(90° - (60° + \theta)) = \cot(60° + \theta)$,

$\therefore \quad \frac{\cos^2(45° + \theta) + \cos^2(45° - \theta)}{\tan(60° + \theta)\tan(60° - \theta)} = \frac{\cos^2(45° + \theta) + \sin^2(45° + \theta)}{\tan(60° + \theta)\cot(60° + \theta)} = \frac{1}{1}$

$(\because \cos^2 A + \sin^2 A = 1$ and $\tan A \cot A = 1)$

$= 1$, as required.

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Example 8. *Prove that:*

(i) (cosec A + cot A) (1 – cos A) = sin A (ii) sec A (1 – sin A) (sec A + tan A) = 1 (NCERT)

Solution. (*i*) LHS = (cosec A + cot A) (1 – cos A)

$$= \left(\frac{1}{\sin A} + \frac{\cos A}{\sin A}\right)(1 - \cos A)$$

$$= \left(\frac{1+\cos A}{\sin A}\right)(1 - \cos A) = \frac{1-\cos^2 A}{\sin A}$$

$$= \frac{\sin^2 A}{\sin A} \qquad (\because 1 - \cos^2 A = \sin^2 A)$$

= sin A = RHS

(*ii*) LHS = sec A (1 – sin A) (sec A + tan A)

$$= \left(\frac{1}{\cos A}\right)(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1-\sin A)(1+\sin A)}{\cos^2 A} = \frac{1-\sin^2 A}{\cos^2 A} = \frac{\cos^2 A}{\cos^2 A} \qquad (\because 1 - \sin^2 A = \cos^2 A)$$

= 1 = RHS

Example 9. *Prove the following:*

(i) $1 + \frac{\cot^2\alpha}{1+\text{cosec}\,\alpha} = \text{cosec}\,\alpha$ (NCERT Exemplar)

(ii) $(\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha) = \sec\alpha + \text{cosec}\,\alpha$ (NCERT Exemplar)

Solution. (*i*) LHS $= 1 + \frac{\cot^2\alpha}{1+\text{cosec}\,\alpha} = 1 + \frac{\text{cosec}^2\alpha - 1}{1+\text{cosec}\,\alpha}$

$$= 1 + \frac{(\text{cosec}\,\alpha + 1)(\text{cosec}\,\alpha - 1)}{1+\text{cosec}\,\alpha} = 1 + (\text{cosec}\,\alpha - 1) = \text{cosec}\,\alpha = \text{RHS}.$$

(*ii*) LHS = $(\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha)$

$$= (\sin\alpha + \cos\alpha)\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right) = (\sin\alpha + \cos\alpha)\frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$= (\sin\alpha + \cos\alpha).\frac{1}{\cos\alpha\sin\alpha} = \frac{\sin\alpha + \cos\alpha}{\cos\alpha\sin\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha\sin\alpha} + \frac{\cos\alpha}{\cos\alpha\sin\alpha} = \frac{1}{\cos\alpha} + \frac{1}{\sin\alpha}$$

$= \sec\alpha + \text{cosec}\,\alpha$ = RHS

Example 10. *Prove the following identities:*

(i) $\frac{\tan A}{1+\sec A} - \frac{\tan A}{1-\sec A} = 2\,\text{cosec}\,A$ (NCERT Exemplar) *(ii)* $\frac{\cot A - \cos A}{\cot A + \cos A} = \frac{\text{cosec}\,A - 1}{\text{cosec}\,A + 1}$ (NCERT)

Solution. (*i*) LHS $= \frac{\tan A}{1+\sec A} - \frac{\tan A}{1-\sec A} = \tan A\left(\frac{1}{1+\sec A} - \frac{1}{1-\sec A}\right)$

$$= \tan A\left(\frac{(1-\sec A)-(1+\sec A)}{1-\sec^2 A}\right) = \tan A\left(\frac{-2\sec A}{-\tan^2 A}\right) \quad (\because \sec^2 A - 1 = \tan^2 A)$$

$$= 2\frac{\sec A}{\tan A} = 2.\frac{1}{\cos A}.\frac{\cos A}{\sin A} = \frac{2}{\sin A} = 2\,\text{cosec}\,A = \text{RHS}$$

(ii) LHS $= \dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\frac{\cos A}{\sin A} - \cos A}{\frac{\cos A}{\sin A} + \cos A} = \dfrac{\cos A\left(\frac{1}{\sin A} - 1\right)}{\cos A\left(\frac{1}{\sin A} + 1\right)} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$ = RHS

Example 11. *Prove the following identities:*

(i) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ (NCERT) (ii) $\dfrac{1+\sec A}{\sec A} = \dfrac{\sin^2 A}{1-\cos A}$ (NCERT)

Solution. (i) LHS $= (\operatorname{cosec}\theta - \cot\theta)^2 = \left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right)^2 = \left(\dfrac{1-\cos\theta}{\sin\theta}\right)^2 = \dfrac{(1-\cos\theta)^2}{\sin^2\theta}$

$= \dfrac{(1-\cos\theta)^2}{1-\cos^2\theta} = \dfrac{(1-\cos\theta)^2}{(1+\cos\theta)(1-\cos\theta)} = \dfrac{1-\cos\theta}{1+\cos\theta}$ = RHS

(ii) LHS $= \dfrac{1+\sec A}{\sec A} = \dfrac{1+\frac{1}{\cos A}}{\frac{1}{\cos A}} = \dfrac{\cos A + 1}{\cos A} \times \dfrac{\cos A}{1}$

$= 1 + \cos A = (1 + \cos A) \times \dfrac{1-\cos A}{1-\cos A}$ (Note this step)

$= \dfrac{1-\cos^2 A}{1-\cos A} = \dfrac{\sin^2 A}{1-\cos A}$ = RHS

Example 12. *Prove the following identities:*

(i) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$ (NCERT Exemplar) (ii) $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$ (NCERT)

Solution. (i) LHS $= \dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)\sin\theta}$

$= \dfrac{\sin^2\theta + 1 + \cos^2\theta + 2\cos\theta}{(1+\cos\theta)\sin\theta} = \dfrac{(\sin^2\theta + \cos^2\theta) + 1 + 2\cos\theta}{(1+\cos\theta)\sin\theta} = \dfrac{1+1+2\cos\theta}{(1+\cos\theta)\sin\theta}$

$= \dfrac{2+2\cos\theta}{(1+\cos\theta)\sin\theta} = \dfrac{2(1+\cos\theta)}{(1+\cos\theta)\sin\theta} = \dfrac{2}{\sin\theta} = 2\operatorname{cosec}\theta$ = RHS

(ii) LHS $= \dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \dfrac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$

$= \dfrac{\sin\theta(1-2(1-\cos^2\theta))}{\cos\theta(2\cos^2\theta-1)} = \dfrac{\sin\theta(2\cos^2\theta - 1)}{\cos\theta(2\cos^2\theta - 1)}$

$= \tan\theta$ = RHS

Example 13. *Prove that:*

(i) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$ (NCERT Exemplar)

(ii) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$ (NCERT)

Solution. (i) LHS $= \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \sqrt{(1+\tan^2\theta)+(1+\cot^2\theta)}$

$= \sqrt{\tan^2\theta + \cot^2\theta + 2} = \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta}$ ($\because \tan\theta\cot\theta = 1$)

$= \sqrt{(\tan\theta + \cot\theta)^2} = \tan\theta + \cot\theta$ = RHS (*ii*) LHS = (cosec A – sin A) (sec A – cos A)

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$$

$$= \frac{1-\sin^2 A}{\sin A} \times \frac{1-\cos^2 A}{\cos A} = \frac{\cos^2 A}{\sin A} \times \frac{\sin^2 A}{\cos A}$$

$$= \cos A \sin A \quad \ldots(1)$$

$$\text{RHS} = \frac{1}{\tan A + \cot A} = \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$$

$$= \frac{1}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}} = \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}$$

$$= \frac{\sin A \cos A}{1} = \sin A \cos A \quad \ldots(2)$$

From (1) and (2), LHS = RHS

Example 14. *Prove the following identities:*

(*i*) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$ (NCERT)

(*ii*) $\tan^4 \theta + \tan^2 \theta = \sec^4 \theta - \sec^2 \theta$ (NCERT Exemplar)

Solution. (*i*) $\text{LHS} = \sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{1+\sin A}{1-\sin A} \times \frac{1+\sin A}{1+\sin A}}$

$$= \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}} = \frac{1+\sin A}{\cos A} = \frac{1}{\cos A} + \frac{\sin A}{\cos A}$$

$$= \sec A + \tan A = \text{RHS}$$

(*ii*) $\tan^4 \theta + \tan^2 \theta = \tan^2 \theta (\tan^2 \theta + 1)$

$$= (\sec^2 \theta - 1) \sec^2 \theta = \sec^4 \theta - \sec^2 \theta = \text{RHS}$$

Example 15. *Prove the following:*

(*i*) $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$ (NCERT)

(*ii*) $(\sin^4 \theta - \cos^4 \theta + 1) \text{cosec}^2 \theta = 1$ (NCERT Exemplar)

Solution. (*i*) LHS = $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2$

$$= \sin^2 A + \text{cosec}^2 A + 2 \sin A \text{ cosec } A + \cos^2 A + \sec^2 A + 2 \cos A \sec A$$

$$= (\sin^2 A + \cos^2 A) + (1 + \cot^2 A) + 2 \times 1 + (1 + \tan^2 A) + 2 \times 1$$

$$= 1 + 6 + \cot^2 A + \tan^2 A = 7 + \tan^2 A + \cot^2 A = \text{RHS}$$

(*ii*) LHS = $(\sin^4 \theta - \cos^4 \theta + 1) \text{cosec}^2 \theta$

$$= ((\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta) + 1) \text{cosec}^2 \theta$$

$$= ((\sin^2 \theta - \cos^2 \theta) \times 1 + 1) \text{cosec}^2 \theta$$

$$= (\sin^2 \theta - \cos^2 \theta + 1) \text{cosec}^2 \theta$$

$$= (\sin^2 \theta + (1 - \cos^2 \theta)) \text{cosec}^2 \theta$$

$$= (\sin^2 \theta + \sin^2 \theta) \text{cosec}^2 \theta$$

$$= 2 \sin^2 \theta \text{ cosec}^2 \theta = 2 (\sin \theta \text{ cosec } \theta)^2$$

$$= 2 \times 1^2 = 2 = \text{RHS}$$

Example 16. *Prove the following:*

(*i*) $\sin^6 \theta + \cos^6 \theta + 3 \sin^2 \theta \cos^2 \theta = 1$ (NCERT Exemplar)

(*ii*) $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec \theta \text{ cosec } \theta$ (NCERT)

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Solution. (*i*) LHS $= \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta)^3 + (\cos^2\theta)^3 + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) + 3\sin^2\theta\cos^2\theta$

$(\because a^3 + b^3 = (a+b)^3 - 3ab(a+b))$

$= (1)^3 - 3\sin^2\theta\cos^2\theta(1) + 3\sin^2\theta\cos^2\theta$

$= 1 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos^2\theta = 1 =$ RHS

(*ii*) LHS $= \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}$

$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}$

$= \frac{1}{\sin\theta-\cos\theta}\left(\frac{\sin^2\theta}{\cos\theta} - \frac{\cos^2\theta}{\sin\theta}\right) = \frac{1}{\sin\theta-\cos\theta} \times \frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta}$

$= \frac{(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)}{(\sin\theta-\cos\theta)\sin\theta\cos\theta} = \frac{1+\sin\theta\cos\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta} + 1 = \sec\theta\,\mathrm{cosec}\,\theta + 1 =$ RHS

Example 17. *Prove that:*

(*i*) $\frac{1+\tan^2 A}{1+\cot^2 A} = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A$ *(NCERT)*

(*ii*) $\frac{1+\sec\theta-\tan\theta}{1+\sec\theta+\tan\theta} = \frac{1-\sin\theta}{\cos\theta}$ *(NCERT Exemplar)*

Solution. (*i*) $\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{\sec^2 A}{\mathrm{cosec}^2 A} = \frac{1}{\cos^2 A} \times \frac{\sin^2 A}{1} = \left(\frac{\sin A}{\cos A}\right)^2 = \tan^2 A.$

Also $\left(\frac{1-\tan A}{1-\cot A}\right)^2 = \left(\frac{1-\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}}\right)^2 = \left(\frac{\cos A-\sin A}{\cos A} \times \frac{\sin A}{\sin A-\cos A}\right)^2$

$= \left(-\frac{\sin A-\cos A}{\cos A} \times \frac{\sin A}{\sin A-\cos A}\right)^2 = \left(-\frac{\sin A}{\cos A}\right)^2 = (-\tan A)^2 = \tan^2 A.$

Hence, $\frac{1+\tan^2 A}{1+\cot^2 A} = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A.$

(*ii*) LHS $= \frac{1+\sec\theta-\tan\theta}{1+\sec\theta+\tan\theta} = \frac{(\sec^2\theta-\tan^2\theta)+(\sec\theta-\tan\theta)}{1+\sec\theta+\tan\theta}$ $(\because \sec^2\theta - \tan^2\theta = 1)$

$= \frac{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)+(\sec\theta-\tan\theta)}{1+\sec\theta+\tan\theta}$

$= \frac{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta+1)}{1+\sec\theta+\tan\theta} = \sec\theta - \tan\theta$

$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \frac{1-\sin\theta}{\cos\theta} =$ RHS

Example 18. *Prove that:*

(*i*) $\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{1}{\sec\theta-\tan\theta}$, *using the identity* $\sec^2\theta = 1+\tan^2\theta$. *(NCERT)*

(*ii*) $\frac{\cos A-\sin A+1}{\cos A+\sin A-1} = \mathrm{cosec}\,A + \cot A$, *using the identity* $\mathrm{cosec}^2 A = 1+\cot^2 A$. *(NCERT)*

Solution. (*i*) Dividing each term of the numerator and denominator of LHS by cos θ, we get

$$\text{LHS} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta} = \frac{(\sec\theta + \tan\theta) - 1}{1 - (\sec\theta - \tan\theta)}$$

$$= \frac{(\sec\theta + \tan\theta) - 1}{(\sec^2\theta - \tan^2\theta) - (\sec\theta - \tan\theta)} \qquad (\because \sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1)$$

$$= \frac{\sec\theta + \tan\theta - 1}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta)}$$

$$= \frac{\sec\theta + \tan\theta - 1}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta - 1)} = \frac{1}{\sec\theta - \tan\theta} = \text{RHS}$$

(*ii*) Dividing each term of the numerator and denominator of LHS by sin A, we get

$$\text{LHS} = \frac{\cot A - 1 + \text{cosec}\, A}{\cot A + 1 - \text{cosec}\, A} = \frac{(\text{cosec}\, A + \cot A) - (\text{cosec}^2 A - \cot^2 A)}{1 + \cot A - \text{cosec}\, A}$$

$$= \frac{(\text{cosec}\, A + \cot A) - (\text{cosec}\, A + \cot A)(\text{cosec}\, A - \cot A)}{1 + \cot A - \text{cosec}\, A}$$

$$= \frac{(\text{cosec}\, A + \cot A)(1 - \text{cosec}\, A + \cot A)}{1 + \cot A - \text{cosec}\, A} = \text{cosec}\, A + \cot A = \text{RHS}$$

Example 19. *Prove that: tan θ + tan (90° – θ) = sec θ sec (90° – θ).* *(NCERT Exemplar)*

Solution. LHS = tan θ + tan (90° – θ) = tan θ + cot θ

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta}$$

= sec θ cosec θ = sec θ sec (90° – θ) = RHS

Example 20. *Given that α + β = 90°, show that*

$$\sqrt{\cos\alpha\,\text{cosec}\,\beta - \cos\alpha\sin\beta} = \sin\alpha$$ *(NCERT Exemplar)*

Solution. Given α + β = 90° ⇒ β = 90° – α ...(*i*)

$$\sqrt{\cos\alpha\,\text{cosec}\,\beta - \cos\alpha\sin\beta} = \sqrt{\cos\alpha\,\text{cosec}(90° - \alpha) - \cos\alpha\sin(90° - \alpha)} \qquad \text{(using (i))}$$

$$= \sqrt{\cos\alpha\sec\alpha - \cos\alpha\cos\alpha} = \sqrt{1 - \cos^2\alpha} \qquad (\because \cos\alpha\sec\alpha = 1)$$

$$= \sqrt{\sin^2\alpha} = \sin\alpha.$$

Example 21. *If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, prove that $\tan\theta = 1$ or $\frac{1}{2}$.* *(NCERT Exemplar)*

Solution. Given $1 + \sin^2\theta = 3\sin\theta\cos\theta$, ...(*i*)

Dividing both sides by $\cos^2\theta$, we get

$$\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = 3\frac{\sin\theta\cos\theta}{\cos^2\theta}$$

⇒ $\sec^2\theta + \tan^2\theta = 3\tan\theta \Rightarrow (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta$

⇒ $2\tan^2\theta - 3\tan\theta + 1 = 0 \Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$

⇒ $2\tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) = 0$

⇒ $(\tan\theta - 1)(2\tan\theta - 1) = 0 \Rightarrow \tan\theta - 1 = 0$ or $2\tan\theta - 1 = 0$

⇒ $\tan\theta = 1$ or $\tan\theta = \frac{1}{2}$.

Example 22. *If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.* *(NCERT Exemplar)*

Solution. Given $\sin\theta + \cos\theta = \sqrt{3}$

⇒ $(\sin\theta + \cos\theta)^2 = (\sqrt{3})^2$ (squaring both sides)

$\Rightarrow$ $(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta = 3$

$\Rightarrow$ $1 + 2\sin\theta\cos\theta = 3 \Rightarrow 2\sin\theta\cos\theta = 2$

$\Rightarrow$ $\sin\theta\cos\theta = 1 \Rightarrow \sin\theta\cos\theta = \sin^2\theta + \cos^2\theta$

$\Rightarrow$ $1 = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$ (Dividing both sides by $\sin\theta\cos\theta$)

$\Rightarrow$ $1 = \dfrac{\sin^2\theta}{\sin\theta\cos\theta} + \dfrac{\cos^2\theta}{\sin\theta\cos\theta} \Rightarrow 1 = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

$\Rightarrow$ $1 = \tan\theta + \cot\theta$.

Example 23. *If $a\sin\theta + b\cos\theta = c$, then prove that*

$a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$ *(NCERT Exemplar)*

Solution. Given $a\sin\theta + b\cos\theta = c$

$\Rightarrow$ $(a\sin\theta + b\cos\theta)^2 = c^2$ (squaring both sides)

$\Rightarrow$ $a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow$ $a^2(1 - \cos^2\theta) + b^2(1 - \sin^2\theta) + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow$ $a^2 + b^2 - (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta) = c^2$

$\Rightarrow$ $a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$

$\Rightarrow$ $(a\cos\theta - b\sin\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow$ $a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

Example 24. *(i) If $\sin\theta + \sin^2\theta = 1$, prove that $\cos^2\theta + \cos^4\theta = 1$.*

(ii) If $\tan^4\theta + \tan^2\theta = 1$, prove that $\cos^4\theta + \cos^2\theta = 1$.

Solution. (*i*) Given $\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow$ $\sin\theta = \cos^2\theta \Rightarrow \sin^2\theta = \cos^4\theta$

$\Rightarrow$ $1 - \cos^2\theta = \cos^4\theta \Rightarrow \cos^2\theta + \cos^4\theta = 1$.

(*ii*) Given $\tan^4\theta + \tan^2\theta = 1 \Rightarrow \tan^2\theta(\tan^2\theta + 1) = 1$

$\Rightarrow$ $1 + \tan^2\theta = \dfrac{1}{\tan^2\theta} \Rightarrow \sec^2\theta = \cot^2\theta$

$\Rightarrow$ $\dfrac{1}{\cos^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} \Rightarrow \sin^2\theta = \cos^4\theta$

$\Rightarrow$ $1 - \cos^2\theta = \cos^4\theta \Rightarrow \cos^4\theta + \cos^2\theta = 1$.

Example 25. *(i) If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show that $\cos\theta - \sin\theta = \sqrt{2}$.*

(ii) If $\sin\theta + 2\cos\theta = 1$, prove that $\cos\theta - 2\sin\theta = 2$. *(NCERT Exemplar)*

Solution. (*i*) Given $\cos\theta + \sin\theta = \sqrt{2}\cos\theta \Rightarrow \sin\theta = (\sqrt{2} - 1)\cos\theta$

$\Rightarrow$ $\cos\theta = \dfrac{1}{\sqrt{2}-1}\sin\theta = \dfrac{1}{\sqrt{2}-1} \times \dfrac{\sqrt{2}+1}{\sqrt{2}+1}\sin\theta = \dfrac{(\sqrt{2}+1)\sin\theta}{2-1}$

$\Rightarrow$ $\cos\theta = \sqrt{2}\sin\theta + \sin\theta \Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

(*ii*) Given $\sin\theta + 2\cos\theta = 1 \Rightarrow 2\cos\theta = 1 - \sin\theta$

$\Rightarrow$ $2\cos\theta(1 + \sin\theta) = (1 - \sin\theta)(1 + \sin\theta)$

$\Rightarrow$ $2\cos\theta(1 + \sin\theta) = 1 - \sin^2\theta$

$\Rightarrow$ $2\cos\theta(1 + \sin\theta) = \cos^2\theta$

$\Rightarrow$ $2(1 + \sin\theta) = \cos\theta$

$\Rightarrow$ $\cos\theta - 2\sin\theta = 2$.

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Example 26. *If $2 \sin^2 \theta - \cos^2 \theta = 2$, then find the value of θ.* *(NCERT Exemplar)*

Solution. Given $2 \sin^2 \theta - \cos^2 \theta = 2$

$\Rightarrow \quad 2 \sin^2 \theta - (1 - \sin^2 \theta) = 2$

$\Rightarrow \quad 3 \sin^2 \theta - 1 = 2 \Rightarrow 3 \sin^2 \theta = 3$

$\Rightarrow \quad \sin^2 \theta = 1 \Rightarrow \sin \theta = 1 \Rightarrow \theta = 90°.$

Hence, the value of θ is $90°$.

Example 27. *(i) If $\tan \theta + \sec \theta = l$, then prove that $\sec \theta = \frac{l^2+1}{2l}$.* *(NCERT Exemplar)*

(ii) If $\operatorname{cosec} \theta + \cot \theta = p$, then prove that $\cos \theta = \frac{p^2-1}{p^2+1}$. *(NCERT Exemplar)*

Solution. (*i*) Given $\tan \theta + \sec \theta = l$ ...(1)

We know that $\sec^2 \theta - \tan^2 \theta = 1$

$\Rightarrow \quad (\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = 1$

$\Rightarrow \quad l(\sec \theta + \tan \theta) = 1$ (using (1))

$\Rightarrow \quad \sec \theta + \tan \theta = \frac{1}{l}$ ...(2)

On adding (1) and (2), we get

$$2 \sec \theta = l + \frac{1}{l} \Rightarrow \sec \theta = \frac{l^2+1}{2l}.$$

(*ii*) Given $\operatorname{cosec} \theta + \cot \theta = p$ ...(1)

We know that $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

$\Rightarrow \quad (\operatorname{cosec} \theta + \cot \theta)(\operatorname{cosec} \theta - \cot \theta) = 1$

$\Rightarrow \quad p(\operatorname{cosec} \theta - \cot \theta) = 1$ (using (1))

$\Rightarrow \quad \operatorname{cosec} \theta - \cot \theta = \frac{1}{p}$ ...(2)

On adding (1) and (2), we get

$$2 \operatorname{cosec} \theta = p + \frac{1}{p} \Rightarrow \operatorname{cosec} \theta = \frac{p^2+1}{2p} \quad \text{...(3)}$$

Subtracting (1) from (2), we get

$$2 \cot \theta = p - \frac{1}{p} \Rightarrow \cot \theta = \frac{p^2-1}{2p} \quad \text{..(4)}$$

Now, $\cos \theta = \frac{\cos \theta}{\sin \theta}.\sin \theta = \cot \theta.\frac{1}{\operatorname{cosec} \theta} = \frac{\cot \theta}{\operatorname{cosec} \theta}$ ...(5)

Dividing (4) by (3), we get

$$\frac{\cot \theta}{\operatorname{cosec} \theta} = \frac{p^2-1}{p^2+1} \Rightarrow \cos \theta = \frac{p^2-1}{p^2+1} \quad \text{(using (5))}$$

Example 28. *(i) If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$, prove that $x^2 + y^2 = a^2 + b^2$.*

(ii) If $x = a \operatorname{cosec} \theta + b \cot \theta$ and $y = a \cot \theta + b \operatorname{cosec} \theta = y$, prove that $x^2 - y^2 = a^2 - b^2$.

Solution. (*i*) Given $x = a \cos \theta + b \sin \theta$ ...(1)

and $\quad y = a \sin \theta - b \cos \theta$ ...(2)

On squaring (1) and (2) and then adding, we get

$$x^2 + y^2 = a^2(\cos^2 \theta + \sin^2 \theta) + b^2(\sin^2 \theta + \cos^2 \theta)$$
$$= a^2 . 1 + b^2 . 1 = a^2 + b^2.$$

(*ii*) Given $x = a \operatorname{cosec} \theta + b \cot \theta$ ...(1)

and $\quad y = a \cot \theta + b \operatorname{cosec} \theta$ ...(2)

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On squaring (1) and (2) and then subtracting, we get

$$x^2 - y^2 = a^2(\operatorname{cosec}^2\theta - \cot^2\theta) + b^2(\cot^2\theta - \operatorname{cosec}^2\theta)$$
$$= a^2 . 1 + b^2(-1) = a^2 - b^2.$$

Example 29. *If $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$, then prove that $q(p^2 - 1) = 2p$.*

(NCERT Exemplar)

Solution. Given $\sin\theta + \cos\theta = p$ ...(*i*) and $\sec\theta + \operatorname{cosec}\theta = q$ ...(*ii*)

$$\Rightarrow \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = q \Rightarrow \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = q$$

$$\Rightarrow \frac{p}{\sin\theta\cos\theta} = q \quad \text{(using (i))}$$

$$\Rightarrow \sin\theta\cos\theta = \frac{p}{q} \quad \text{...(iii)}$$

On squaring (*i*), we get

$$(\sin\theta + \cos\theta)^2 = p^2 = (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta = p^2$$

$$\Rightarrow 1 + 2\frac{p}{q} = p^2 \quad \text{(using (iii))}$$

$$\Rightarrow \frac{2p}{q} = p^2 - 1 \Rightarrow 2p = q(p^2 - 1).$$

Example 30. *If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then prove that $m^2 - n^2 = 4\sqrt{mn}$.*

Solution. Given $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$

$$\Rightarrow \tan\theta = \frac{m+n}{2} \text{ and } \sin\theta = \frac{m-n}{2}$$

$$\Rightarrow \cot\theta = \frac{2}{m+n} \text{ and } \operatorname{cosec}\theta = \frac{2}{m-n}.$$

Now using $\operatorname{cosec}^2\theta - \cot^2\theta = 1$, we get

$$\left(\frac{2}{m-n}\right)^2 - \left(\frac{2}{m+n}\right)^2 = 1$$

$$\Rightarrow 4[(m+n)^2 - (m-n)^2] = (m-n)^2(m+n)^2$$

$$\Rightarrow 4 \times 4mn = (m^2 - n^2)^2 \Rightarrow m^2 - n^2 = 4\sqrt{mn}.$$

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